Cone-beam reconstruction by the use of Radon transform intermediate functions

R. Clack

Medical Imaging Research Laboratory, Department of Radiology, University of Utah, Salt Lake City, Utah 84132

M. Defrise

Division of Nuclear Medicine, Free University of Brussels, AZ-VUB, Laarbeeklaan 101, B-1090 Brussels, Belgium

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A mathematical framework is presented for cone-beam reconstruction by intermediate functions that are related to the three-dimensional Radon transform of the object being imaged. Cone-beam projection data are processed with a filter to form an intermediate function. The filter is a linear combination $ah_1(l) + bh_2(l)$ of the ramp kernel $h_1(l)$ and the derivative functional $h_2(l)$. From the intermediate function, the reconstruction is completed with a convolution and backprojection, where the convolution filter is another linear combination $ch_1(l) + dh_2(l)$ subject to the restriction $ac + bd = 1$. This formulation unifies and generalizes the important cone-beam formulas of Tuy [SIAM J. Appl. Math. **43,** 546 (1983)], Smith [IEEE Trans. Med. Imaging **MI-4,** 14 (1985)], and Grangeat [Ph.D. dissertation (Ecole Nationale Supérieure des Télécommunications, Paris, 1987)]. The appropriate values of a, b, c, d for these methods are derived.

1. INTRODUCTION AND GENERAL INVERSION FORMULA

Reconstruction of functions from their cone-beam projections is a difficult problem that has received increasing attention in the past few years. The first theoretical results appeared in 1961, when Kirillov[1] published a short mathematical note on cone-beam reconstruction in n-dimensional complex space. In three dimensions, Kirillov's formulas require that the path of the cone vertex (the orbit) satisfy the condition that it is intersected by every plane in space. Twenty years later, formulas based partially on Kirillov's research but adapted for a medical-imaging setting began to appear. Tuy[2] and Smith[3,4] both presented inversion formulas and more practical orbit conditions. Working independently, Grangeat[5,6] also derived an inversion formula and the same orbit conditions. Although a number of approximate algorithms have been reported, only a few implementations of algorithms capable of arbitrarily precise reconstruction (exact reconstruction) have appeared,[7–10] and these implementations have all been based directly on these earlier theoretical papers. Although the formulas of Tuy, Smith, and Grangeat all involve some link to the three-dimensional (3D) Radon transform of the object, to our knowledge a thorough examination of the differences and overlap of the theories has never been published. The purpose of this paper is to provide a simpler and unified derivation of the important results of Tuy, Smith, and Grangeat.

Other researchers have commented on relationships between these results. Grangeat[5] uses Tuy's lemma to illustrate that the first derivative of the Radon transform can be obtained from the cone-beam data. In both Refs. 6 and 8 it is pointed out that Smith's intermediate function is related to Grangeat's by a Hilbert transform.

In this paper we present a cone-beam inversion formula that incorporates the important theoretical results of Tuy,[2] Smith,[3,4] and Grangeat.[5,6] The formula represents a class of mathematically exact reconstruction procedures and applies to any cone-beam system that collects complete (not truncated) projections for orbits satisfying Kirillov's condition. With some modification of the limits of integration, the formula can be adapted to apply to the more practical orbits that satisfy Tuy's condition. The formula uses four parameters (a, b, c, d) to incorporate general intermediate functions that link the cone-beam projection data to the 3D Radon transform and to incorporate a general convolution and backprojection step to process the intermediate functions. Different values of the parameters give different inversion formulas, with $(a, b, c, d) = (1/2, i/2, 0, -2i)$, $(a, b, c, d) = (1, 0, 1, 0)$, and $(a, b, c, d) = (0, -2\pi, 0, -1/2\pi)$ yielding methods (equivalent to those) of Tuy, Smith, and Grangeat, respectively. Any assignment of (a, b, c, d) satisfying $ac + bd = 1$ gives a valid inversion procedure, but because of some scaling properties of the parameters not all the inversion procedures will be mathematically distinct.

The reconstruction formula is given below, and in Section 2 an overview of the derivation is presented. More details of the derivation can be found in Appendix A. In Sections 3, 4, and 5 the parameter values corresponding to the methods of Tuy, Smith, and Grangeat, respectively, are derived. In Section 6 a brief discussion shows how the methods diverge from the common framework when the classical Tuy orbit condition is applied. Finally, the parameters a, b, c, d are reduced to a single parameter γ by removing redundancies in the reconstruction formula.

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We assume that the function to be reconstructed, $f(\mathbf{x})$, is sufficiently smooth on its domain and is supported inside some ball of radius R. The orbit is some almost everywhere differentiable curve $\mathbf{\Phi}(\Lambda)$ in space, parameterized by $\lambda \in \Lambda \subseteq \mathbb{R}$, where Λ is an interval and $\mathbb{R}$ is the set of real numbers. Points on the curve are given by $\mathbf{\Phi}(\lambda)$. The orbit is assumed to satisfy Kirillov's condition that any plane in space intersects the orbit. Mathematically stated, the condition requires that the curve $\mathbf{\Phi}(\Lambda)$ have the property that for all $\boldsymbol{\theta} \in S^2$ (the set of unit vectors in $\mathbb{R}^3$), and all $l \in \mathbb{R}$, there exists λ such that $\mathbf{\Phi}(\lambda) \cdot \boldsymbol{\theta} = l$.

The cone-beam projection data are represented by the function g, defined on $S^2 \times \Lambda$. The value of $g(\boldsymbol{\theta}, \lambda)$ represents the integral along the half-line from the orbit point $\mathbf{\Phi}(\lambda)$ in the direction $\boldsymbol{\theta}$,

$$g(\boldsymbol{\theta}, \lambda) = \int_0^\infty f(\mathbf{\Phi}(\lambda) + t\boldsymbol{\theta})\mathrm{d}t. \tag{1}$$

For any complex scalars a and b, an intermediate function G also defined on $S^2 \times \Lambda$ is given by

$$G(\boldsymbol{\beta}, \lambda) = \int_{S^2} g(\boldsymbol{\theta}, \lambda)[ah_1 + bh_2](\boldsymbol{\theta} \cdot \boldsymbol{\beta})\mathrm{d}\boldsymbol{\theta}, \tag{2}$$

where h_1 and h_2 are two real generalized functions defined by the inverse Fourier transforms

$$h_1(t) = \int_{-\infty}^\infty |\omega|\exp(2\pi i\omega t)\mathrm{d}\omega,$$

$$h_2(t) = i\int_{-\infty}^\infty \omega \exp(2\pi i\omega t)\mathrm{d}\omega \tag{3}$$

and a linear combination such as $[ah_2 + bh_2](t)$ is defined as the inverse Fourier transform of $a|\omega| + ib|\omega|$. These two functions are more familiarly known as the tomographic ramp-filter kernel (h_1) and the derivative of the Dirac delta function [$h_2(t) = \delta'(t)/2\pi$]. Note that convolution by $2\pi h_2$ corresponds to taking a derivative.

The reconstruction formula can now be given as follows. For any complex scalars c and d such that $ac + bd = 1$,

$$f(\mathbf{x}) = \frac{1}{2}\int_{S^2} F^*(\boldsymbol{\beta}, \mathbf{x} \cdot \boldsymbol{\beta})\mathrm{d}\boldsymbol{\beta}, \tag{4}$$

where

$$F^*(\boldsymbol{\beta}, l) = \int_{-\infty}^\infty F(\boldsymbol{\beta}, t)[ch_1 + dh_2](l - t)\mathrm{d}t, \tag{5}$$

and, for all $\boldsymbol{\beta}$ and l, $F(\boldsymbol{\beta}, l)$ is given by the rebinning equation

$$F(\boldsymbol{\beta}, l) = G(\boldsymbol{\beta}, \lambda), \tag{6}$$

where λ is chosen to be a solution (ensured to exist by Kirillov's condition) to the equation $\mathbf{\Phi}(\lambda) \cdot \boldsymbol{\beta} = l$.

2. DERIVATION OF THE FORMULA

We let $r(\boldsymbol{\beta}, l)$ be the 3D Radon transform of $f(\mathbf{x})$:

$$r(\boldsymbol{\beta}, l) = \int_{\mathbb{R}^3} f(\mathbf{x})\delta(\mathbf{x} \cdot \boldsymbol{\beta} - l)\mathrm{d}\mathbf{x}. \tag{7}$$

The derivation proceeds by a sequence of lemmas. The lemmas are listed below, and their formal derivations may be found in Appendix A. Lemma 1 describes the link between the intermediate function $F(\boldsymbol{\beta}, l)$ and the Radon transform $r(\boldsymbol{\beta}, l)$. Specific cases of lemma 1 have been established by Tuy [equation (8) in Ref. 2—Tuy's lemma], Smith [equation (3.8) in Ref. 3], and Grangeat [equation (3.23) in Ref. 5]. Lemma 2 is the 3D Radon inversion formula expressed with h_1 or h_2. Lemma 2 is used for the proof of lemma 3, and the reconstruction procedure of Eqs. (2) and (4)–(6) follows from lemmas 1 and 3.

Lemma 1: If $G(\boldsymbol{\beta}, \lambda)$ is given by Eq. (2),

$$G(\boldsymbol{\beta}, \lambda) = \int_{S^2} g(\boldsymbol{\theta}, \lambda)[ah_1 + bh_2](\boldsymbol{\theta} \cdot \boldsymbol{\beta})\mathrm{d}\boldsymbol{\theta},$$

then $G(\boldsymbol{\beta}, \lambda) = F(\boldsymbol{\beta}, \mathbf{\Phi}(\lambda) \cdot \boldsymbol{\beta})$ if and only if

$$F(\boldsymbol{\beta}, l) = \int_{-\infty}^\infty r(\boldsymbol{\beta}, t)[ah_1 - bh_2](l - t)\mathrm{d}t, \tag{8}$$

where $g(\boldsymbol{\theta}, \lambda)$ and $r(\boldsymbol{\beta}, t)$ are given by Eqs. (1) and (7), respectively. ■

We write convolutions with an $*$, so with a slight abuse of notation, Eq. (8) can be written as $F(\boldsymbol{\beta}, l) = r(\boldsymbol{\beta}, l) * (ah_1 - bh_2)(l)$.

Lemma 2:

$$f(\mathbf{x}) = \frac{1}{2}\int_{S^2} [r(\boldsymbol{\beta}, l) * h_1(l) * h_1(l)]_{l=\mathbf{x}\cdot\boldsymbol{\beta}}\mathrm{d}\boldsymbol{\beta}, \tag{9}$$

$$f(\mathbf{x}) = -\frac{1}{2}\int_{S^2} [r(\boldsymbol{\beta}, l) * h_2(l) * h_2(l)]_{l=\mathbf{x}\cdot\boldsymbol{\beta}}\mathrm{d}\boldsymbol{\beta}. \tag{10}$$

■

Lemma 3: If

$$F^*(\boldsymbol{\beta}, l) = r(\boldsymbol{\beta}, l) * [ah_1 - bh_2](l) * [ch_1 + dh_2](l), \tag{11}$$

then

$$f(\mathbf{x}) = \frac{1}{2(ac + bd)}\int_{S^2} F^*(\boldsymbol{\beta}, \mathbf{x} \cdot \boldsymbol{\beta})\mathrm{d}\boldsymbol{\beta}, \tag{12}$$

where $r(\boldsymbol{\beta}, l)$ is given by Eq. (7). ■

The reconstruction formula now follows easily. By lemma 1, for $F(\boldsymbol{\beta}, l)$ given by Eqs. (2) and (6) we obtain $F(\boldsymbol{\beta}, l) = r(\boldsymbol{\beta}, l) * [ah_1 - bh_2](l)$, and hence $F^*(\boldsymbol{\beta}, l)$ given by Eq. (5) satisfies the hypothesis of lemma 3. Thus, by lemma 3, for $ac + bd = 1$ reconstruction is achieved by the backprojection of F^*, as given in Eq. (4).

3. TUY'S FORMULA

First we show that $(a, b) = (1/2, i/2)$ corresponds to Tuy's intermediate function G^T. Tuy's intermediate function was obtained by extending g [as defined in Eq. (1)] to the domain $\mathbb{R}^3 \times \Lambda$ by positive homogeneity, then taking the 3D Fourier transform. Thus, by writing g^T for Tuy's definition of the cone-beam data, we obtain

$$g^T(\mathbf{x},\lambda) = \frac{g(\mathbf{x}/|\mathbf{x}|,\lambda)}{|\mathbf{x}|}, \tag{13}$$

$$\begin{aligned} G^T(\boldsymbol{\xi},\lambda) &= \int_{\mathbb{R}^3} g^T(\mathbf{x},\lambda)\exp(-2\pi i\mathbf{x}\cdot\boldsymbol{\xi})\mathrm{d}\mathbf{x} \\ &= \int_{S^2}\int_0^\infty g^T(t\boldsymbol{\theta},\lambda)\exp(-2\pi i t\boldsymbol{\theta}\cdot\boldsymbol{\xi})t^2\mathrm{d}t\mathrm{d}\boldsymbol{\theta} \\ &= \int_{S^2}\int_0^\infty g^T(\boldsymbol{\theta},\lambda)\exp(-2\pi i t\boldsymbol{\theta}\cdot\boldsymbol{\xi})t\mathrm{d}t\mathrm{d}\boldsymbol{\theta} \\ &= \int_{S^2} g(\boldsymbol{\theta},\lambda)\int_{-\infty}^\infty \frac{|t|+t}{2}\exp(-2\pi i t\boldsymbol{\theta}\cdot\boldsymbol{\xi})\mathrm{d}t\mathrm{d}\boldsymbol{\theta} \\ &= \int_{S^2} g(\boldsymbol{\theta},\lambda)\int_{-\infty}^\infty \frac{|t|-t}{2}\exp(2\pi i t\boldsymbol{\theta}\cdot\boldsymbol{\xi})\mathrm{d}t\mathrm{d}\boldsymbol{\theta} \\ &= \int_{S^2} g(\boldsymbol{\theta},\lambda)\left[\frac{h_1}{2}-\frac{h_2}{2i}\right](\boldsymbol{\theta}\cdot\boldsymbol{\xi})\mathrm{d}\boldsymbol{\theta}. \end{aligned} \tag{14}$$

Therefore $a = 1/2$, $b = i/2$. By lemma 1, $F(\boldsymbol{\beta}, l) = r(\boldsymbol{\beta}, l) * [(h_1 - ih_2)/2](l)$, where $F(\boldsymbol{\beta}, \boldsymbol{\Phi}(\lambda)\cdot\boldsymbol{\beta}) = G^T(\boldsymbol{\beta},\lambda)$ according to Eq. (6). This expression for F is easily seen to correspond to Tuy's lemma.[2]

Of the range of possible choices of (c, d) that satisfy $(1/2)c + (i/2)d = 1$, Tuy's formula corresponds to $(c, d) = (0, -2i)$, because then we have

$$\begin{aligned} f(\mathbf{x}) &= \frac{1}{2}\int_{S^2}\int_{-\infty}^\infty F(\boldsymbol{\beta},l)(-2i)h_2(\mathbf{x}\cdot\boldsymbol{\beta}-l)\mathrm{d}l\mathrm{d}\boldsymbol{\beta}, \\ &= \frac{1}{2\pi i}\int_{S^2}\left.\frac{\partial F(\boldsymbol{\beta},l)}{\partial l}\right|_{l=\mathbf{x}\cdot\boldsymbol{\beta}}\mathrm{d}\boldsymbol{\beta}. \end{aligned} \tag{15}$$

Now because

$$\frac{\partial G(\boldsymbol{\beta},\lambda)}{\partial\lambda} = \frac{\partial F(\boldsymbol{\beta},\boldsymbol{\Phi}(\lambda)\cdot\boldsymbol{\beta})}{\partial\lambda} = (\boldsymbol{\Phi}'(\lambda)\cdot\boldsymbol{\beta})\left.\frac{\partial F(\boldsymbol{\beta},l)}{\partial l}\right|_{l=\boldsymbol{\Phi}(\lambda)\cdot\boldsymbol{\beta}}, \tag{16}$$

we obtain

$$f(\mathbf{x}) = \frac{1}{2\pi i}\int_{S^2}\frac{1}{\boldsymbol{\Phi}'(\lambda)\cdot\boldsymbol{\beta}}\frac{\partial G(\boldsymbol{\beta},\lambda)}{\partial\lambda}\mathrm{d}\boldsymbol{\beta}, \tag{17}$$

where λ is chosen such that $\mathbf{x}\cdot\boldsymbol{\beta} = \boldsymbol{\Phi}(\lambda)\cdot\boldsymbol{\beta}$ and $\boldsymbol{\Phi}'(\lambda)\cdot\boldsymbol{\beta} \neq 0$. Note that this last condition implies a slightly more restrictive orbit condition, that is, that every plane must have some nontangential intersection with the orbit.

Equation (17) is Tuy's formula [equation (9) in Ref. 2].

4. SMITH'S FORMULA

We show that $(a, b, c, d) = (1, 0, 1, 0)$ for the results of Smith. Smith uses the same technique of extension to $\mathbb{R}^3$ and taking of the 3D Fourier-transform function as does Tuy. The critical difference is the initial definition of g; Smith starts with a function that is even in $\boldsymbol{\theta}$. If g^S and G^S are used to denote the functions of Smith's derivation,

$$g^S(\boldsymbol{\theta},\lambda) = \int_{-\infty}^\infty f(\boldsymbol{\Phi}(\lambda)+t\boldsymbol{\theta})\mathrm{d}t = g(\boldsymbol{\theta},\lambda)+g(-\boldsymbol{\theta},\lambda), \tag{18}$$

$$g^S(\mathbf{x},\lambda) = \frac{g^S(\mathbf{x}/|\mathbf{x}|,\lambda)}{|\mathbf{x}|} = \frac{g(\mathbf{x}/|\mathbf{x}|,\lambda)+g(-\mathbf{x}/|\mathbf{x}|,\lambda)}{|\mathbf{x}|}, \tag{19}$$

$$\begin{aligned} G^S(\boldsymbol{\beta},\lambda) &= \int_{\mathbb{R}^3} g^S(\mathbf{x},\lambda)\exp(-2\pi i\mathbf{x}\cdot\boldsymbol{\beta})\mathrm{d}\mathbf{x} \\ &= \int_{S^2}(g(\boldsymbol{\theta},\lambda)+g(-\boldsymbol{\theta},\lambda))\left[\frac{h_1+ih_2}{2}\right](\boldsymbol{\theta}\cdot\boldsymbol{\beta})\mathrm{d}\boldsymbol{\theta}. \end{aligned} \tag{20}$$

The last step follows from the same sequence of manipulations as in Section 3. Using the even (odd) property of h_1 (h_2), we obtain

$$\int_{S^2} g(-\boldsymbol{\theta},\lambda)\left[\frac{h_1+ih_2}{2}\right](\boldsymbol{\theta}\cdot\boldsymbol{\beta})\mathrm{d}\boldsymbol{\theta} = \int_{S^2} g(\boldsymbol{\theta},\lambda)\left[\frac{h_1-ih_2}{2}\right](\boldsymbol{\theta}\cdot\boldsymbol{\beta})\mathrm{d}\boldsymbol{\theta}, \tag{21}$$

which, when substituted into the expression for $G^S(\boldsymbol{\beta},\lambda)$, yields

$$\begin{aligned} G^S(\boldsymbol{\beta},\lambda) &= \int_{S^2} g(\boldsymbol{\theta},\lambda)\left[\frac{h_1+ih_2}{2}\right](\boldsymbol{\theta}\cdot\boldsymbol{\beta}) + g(\boldsymbol{\theta},\lambda)\left[\frac{h_1-ih_2}{2}\right](\boldsymbol{\theta}\cdot\boldsymbol{\beta})\mathrm{d}\boldsymbol{\theta}. \\ &= \int_{S^2} g(\boldsymbol{\theta},\lambda)h_1(\boldsymbol{\theta}\cdot\boldsymbol{\beta})\mathrm{d}\boldsymbol{\theta}, \end{aligned} \tag{22}$$

by which we establish that $(a, b) = (1, 0)$ for Smith's intermediate function. By lemma 1, $F(\boldsymbol{\beta}, l) = r(\boldsymbol{\beta}, l) * h_1(l)$, where $F(\boldsymbol{\beta}, \boldsymbol{\Phi}(\lambda)\cdot\boldsymbol{\beta}) = G^S(\boldsymbol{\beta}, \lambda)$. This result was demonstrated directly in Smith,[3] although some factors of 2π differ in his result because of a different definition of the Fourier transform.

Smith's reconstruction formula [see equation (2.5) in Ref. 3] was given as

$$f(\mathbf{x}) = \int_{S^2/2} F(\boldsymbol{\beta},l) * h_1(l)|_{l=\mathbf{x}\cdot\boldsymbol{\beta}}\mathrm{d}\boldsymbol{\beta}, \tag{23}$$

where $S^2/2$ is some fixed hemisphere. We show that the integrand $F(\boldsymbol{\beta}, l) * h_1(l)|_{l=\mathbf{x}\cdot\boldsymbol{\beta}}$ is even in $\boldsymbol{\beta}$ as follows. The notation $\check{r}$ refers to the Fourier transform of r with respect to the second variable. It is easily verified that $\check{r}(-\boldsymbol{\beta}, -\omega) = \check{r}(\boldsymbol{\beta}, \omega)$ (see also the last line of the proof of lemma 3), so

$$\begin{aligned} F(\boldsymbol{\beta},l) * h_1(l)|_{l=\mathbf{x}\cdot\boldsymbol{\beta}} &= r(\boldsymbol{\beta},l) * h_1(l) * h_1(l)|_{l=\mathbf{x}\cdot\boldsymbol{\beta}} \\ &= \int_{-\infty}^\infty \check{r}(\boldsymbol{\beta},\omega)|\omega|\,|\omega|\exp(2\pi i\omega\mathbf{x}\cdot\boldsymbol{\beta})\mathrm{d}\omega \\ &= \int_{-\infty}^\infty \check{r}(-\boldsymbol{\beta},-\omega)\omega^2\exp(2\pi i\omega\mathbf{x}\cdot\boldsymbol{\beta})\mathrm{d}\omega \\ &= \int_{-\infty}^\infty \check{r}(-\boldsymbol{\beta},\omega)\omega^2\exp(-2\pi i\omega\mathbf{x}\cdot\boldsymbol{\beta})\mathrm{d}\omega \\ &= F(-\boldsymbol{\beta},l) * h_1(l)|_{l=-\mathbf{x}\cdot\boldsymbol{\beta}}. \end{aligned} \tag{24}$$

Therefore the integration over $S^2/2$ in Eq. (23) can be extended to S^2 to yield

$$f(x) = \frac{1}{2}\int_{S^2} F(\boldsymbol{\beta}, l) * h_1(l)|_{l=\mathbf{x}\cdot\boldsymbol{\beta}} \mathrm{d}\boldsymbol{\beta} = \frac{1}{2}\int_{S^2} F^*(\boldsymbol{\beta}, \mathbf{x}\cdot\boldsymbol{\beta})\mathrm{d}\boldsymbol{\beta}, \tag{25}$$

where $F^*(\boldsymbol{\beta}, l) = F(\boldsymbol{\beta}, l) * h_1(l)$ and we establish $(c, d) = (1, 0)$.

5. GRANGEAT'S FORMULA

We show here that for Grangeat's formulation, $(a, b, c, d) = (0, -2\pi, 0, -1/2\pi)$. In section 2.3.3 of Ref. 5, Grangeat derives his results in terms of a flat detector coordinate system. It can be shown by making a suitable change of variables[10] that his intermediate function G^G corresponds to

$$G^G(\boldsymbol{\beta}, \lambda) = \int_{S^2} g(\boldsymbol{\theta}, \lambda)(-2\pi)h_2(\boldsymbol{\theta}\cdot\boldsymbol{\beta})\mathrm{d}\boldsymbol{\theta}, \tag{26}$$

which establishes that $(a, b) = (0, -2\pi)$ and $F(\boldsymbol{\beta}, l) = r(\boldsymbol{\beta}, l) * 2\pi h_2(l) = (\partial/\partial l)r(\boldsymbol{\beta}, l)$, where $F(\boldsymbol{\beta}, \boldsymbol{\Phi}(\lambda)\cdot\boldsymbol{\beta}) = G^G(\boldsymbol{\beta}, \lambda)$. This equation is precisely Grangeat's result on the first derivative of the Radon transform.[5,6]

Grangeat's reconstruction proceeds as follows:

$$\begin{aligned} f(\mathbf{x}) &= \frac{-1}{8\pi^2}\int_{S^2} \frac{\partial}{\partial l}\frac{\partial r(\boldsymbol{\beta}, l)}{\partial l}\bigg|_{l=\mathbf{x}\cdot\boldsymbol{\beta}} \mathrm{d}\boldsymbol{\beta} \\ &= \frac{-1}{8\pi^2}\int_{S^2} \frac{\partial F(\boldsymbol{\beta}, l)}{\partial l}\bigg|_{l=\mathbf{x}\cdot\boldsymbol{\beta}} \mathrm{d}\boldsymbol{\beta} \\ &= \frac{-1}{8\pi^2}\int_{S^2} F(\boldsymbol{\beta}, l) * 2\pi h_2(l)|_{l=\mathbf{x}\cdot\boldsymbol{\beta}} \mathrm{d}\boldsymbol{\beta} \\ &= \frac{1}{2}\int_{S^2} F(\boldsymbol{\beta}, l) * \frac{-1}{2\pi}h_2(l)|_{l=\mathbf{x}\cdot\boldsymbol{\beta}} \mathrm{d}\boldsymbol{\beta}, \end{aligned} \tag{27}$$

which shows that $(c, d) = (0, -1/2\pi)$ as claimed.

6. ORBIT CONDITIONS AND DISCUSSION

To maintain a unified presentation, the Kirillov orbit condition has been assumed. In fact a much less restrictive condition can be found at the cost of a more involved reconstruction formula. We confine our attention here to the cases that include the formulas of Tuy, Smith, and Grangeat.

The Kirillov condition ensures that for all $\boldsymbol{\beta} \in S^2$ and all $l \in [-\infty, \infty]$, the value of $F(\boldsymbol{\beta}, l)$ is available for the integrand of Eq. (5), using the rebinning step of Eq. (6). In fact reconstruction is still possible when $F(\boldsymbol{\beta}, l)$ is available for all $\boldsymbol{\beta}$ and the restricted range $|l| < R$, where R is the radius of the support of $f(\mathbf{x})$. The significance of the restricted range is that the orbit only need satisfy the condition that just those planes intersecting the support of $f(\mathbf{x})$ are required to intersect the orbit. This improved orbit condition, usually referred to as Tuy's sufficiency condition, was derived in Refs. 2, 3, and 5.

For orbits satisfying Tuy's condition, the reconstruction formula is more complicated. We consider here three special cases, which encompass the formulas of Grangeat, Tuy, and Smith, respectively.

If $a = 0$, then it can be shown that $F(\boldsymbol{\beta}, l) = 0$ for $|l| > R$ as follows. Because $f(\mathbf{x}) = 0$ for $|\mathbf{x}| > R$, it can be seen that the 3D Radon transform, $r(\boldsymbol{\beta}, l)$ must be zero for $|l| > R$, and from there it follows trivially that $\partial r/\partial l = 0$ for $|l| > R$. By Eq. (8), for $a = 0$ we have $F(\boldsymbol{\beta}, l) = r(\boldsymbol{\beta}, l) * (-b)h_2(l) = (-b/2\pi)\partial r/\partial l$, and hence $F(\boldsymbol{\beta}, l)$ vanishes for $|l| > R$. Therefore Eq. (5) can be replaced by

$$F^*(\boldsymbol{\beta}, l) = \int_{-R}^{R} F(\boldsymbol{\beta}, t)[ch_1 + dh_2](l - t)\mathrm{d}t. \tag{5'}$$

The second special case is $c = 0$. We observe that only values of $f(\mathbf{x})$ for $|\mathbf{x}| \le R$ are of interest, and because for any unit vector $\boldsymbol{\beta}$, $|\mathbf{x}\cdot\boldsymbol{\beta}| \le |\mathbf{x}| \le R$ we have from Eq. (4) that $F^*(\boldsymbol{\beta}, l)$ is only required for $|l| \le R$. Now for $c = 0$, Eq. (5) reduces to $F^*(\boldsymbol{\beta}, l) = (d/2\pi)\partial F/\partial l$, and thus although $F(\boldsymbol{\beta}, l)$ might be nonzero outside $|l| < R$, it is only needed in this restricted range. Again, Eq. (5) can be replaced by Eq. (5′).

The third case we consider is $b = d = 0$. Here the situation is more complicated, and we refer the reader to the discussion in sections IV and V of Ref. 3 and to Ref. 11 in this paper. Again it is true that $F(\boldsymbol{\beta}, l)$ is only needed in the range $|l| < R$. Equation (5′) does not hold, however; the integration range varies with $\boldsymbol{\beta}$ and $\mathbf{x}$, and Eq. (5) is replaced with

$$F^*(\boldsymbol{\beta}, l) = \int_{L_1}^{L_2} F(\boldsymbol{\beta}, t)ch_1(l - t)\mathrm{d}t, \tag{5''}$$

where for any fixed unit vector $\boldsymbol{\zeta}$, the integration limits L_1 and L_2 may be defined as follows:

$$L_1 = -(R^2 - (\mathbf{x}\cdot\boldsymbol{\zeta})^2)^{1/2}(1 - (\boldsymbol{\beta}\cdot\boldsymbol{\zeta})^2)^{1/2} + (\mathbf{x}\cdot\boldsymbol{\zeta})(\boldsymbol{\beta}\cdot\boldsymbol{\zeta}), \tag{28}$$

$$L_2 = (R^2 - (\mathbf{x}\cdot\boldsymbol{\zeta})^2)^{1/2}(1 - (\boldsymbol{\beta}\cdot\boldsymbol{\zeta})^2)^{1/2} + (\mathbf{x}\cdot\boldsymbol{\zeta})(\boldsymbol{\beta}\cdot\boldsymbol{\zeta}). \tag{29}$$

It can be shown that $0 \le |L_1| \le R$ and $0 \le |L_2| \le R$.

The last point that we discuss involves the number of degrees of freedom that are available in the complex parameters (a, b, c, d). A scaling of Eq. (2) by an arbitrary factor k can be offset in Eq. (5). Thus the parameters (a, b, c, d) and $(ka, kb, c/k, d/k)$ clearly represent equivalent inversion procedures. This flexibility can be removed by insisting that the first nonzero entry of the four-tuple be equal to 1. A second restriction, $ac + bd = 1$ has already been introduced. Finally, we remark that the integration over S^2 in Eq. (4) includes a cancellation effect. Specifically, the odd part of $F^*(\boldsymbol{\beta}, \mathbf{x}\cdot\boldsymbol{\beta})$ makes no contribution to the reconstruction. An examination of the proof of lemma 3 reveals that the odd part of $F^*(\boldsymbol{\beta}, l)$ is $r(\boldsymbol{\beta}, l) * [(ad - bc)(h_1 * h_2)](l)$, and we can force $F^*(\boldsymbol{\beta}, l)$ to be even [which allows Eq. (4) to be written as $f(\mathbf{x}) = \int_{S^2/2} F^*(\boldsymbol{\beta}, \mathbf{x}\cdot\boldsymbol{\beta})\mathrm{d}\boldsymbol{\beta}$ for some fixed hemisphere $S^2/2$] by requiring $ad - bc = 0$.

When these constraints on a, b, c, d are combined, the range of parameters reduces to $(a, b, c, d) = (0, 1, 0, 1)$ or $(1, \gamma, 1/(1 + \gamma^2), \gamma/(1 + \gamma^2))$ for arbitrary complex γ not equal to $\pm i$. For more insight into the significance of the parameter γ, we define for some fixed orbit $\boldsymbol{\Phi}(\Lambda)$ and given projection data $g(\boldsymbol{\theta}, \lambda)$ the following quantities. For

$(a,b,c,d) = (1,0,1,0)$, let $G_S(\boldsymbol{\beta},\lambda)$, $F_S^*(\boldsymbol{\beta},l)$, and $f_S(\mathbf{x})$ be given by Eqs. (2), (5), and (4), using $F_S(\boldsymbol{\beta},\boldsymbol{\Phi}(\lambda)\cdot\boldsymbol{\beta}) = G_S(\boldsymbol{\beta},\lambda)$. For $(a,b,c,d) = (0,-2\pi,0,-1/2\pi)$, define $G_G(\boldsymbol{\beta},\lambda)$, $F_G^*(\boldsymbol{\beta},l)$, $f_G(\mathbf{x})$, and $F_G(\boldsymbol{\beta},l)$ similarly. For $(a,b,c,d) = (1,\gamma,1/(1+\gamma^2),\gamma/(1+\gamma^2))$, we obtain

$$G(\boldsymbol{\beta},\lambda) = G_S(\boldsymbol{\beta},\lambda) + \frac{\gamma}{-2\pi}G_G(\boldsymbol{\beta},\lambda), \tag{30}$$

$$F(\boldsymbol{\beta},l) = F_S(\boldsymbol{\beta},l) + \frac{\gamma}{-2\pi}F_G(\boldsymbol{\beta},l), \tag{31}$$

$$\begin{aligned} F^*(\boldsymbol{\beta},l) = {} & \frac{1}{1+\gamma^2}F_S^*(\boldsymbol{\beta},l) + \frac{\gamma^2}{1+\gamma^2}F_G^*(\boldsymbol{\beta},l) \\ & + \frac{\gamma}{1+\gamma^2}F_S(\boldsymbol{\beta},l) * h_2(l) \\ & + \frac{\gamma}{-2\pi(1+\gamma^2)}F_G(\boldsymbol{\beta},l) * h_1(l). \end{aligned} \tag{32}$$

It can be shown that the last two terms of $F^*(\boldsymbol{\beta},\mathbf{x}\cdot\boldsymbol{\beta})$ are odd in $\boldsymbol{\beta}$, and therefore

$$f(\mathbf{x}) = \frac{1}{1+\gamma^2}f_S(\mathbf{x}) + \frac{\gamma^2}{1+\gamma^2}f_G(\mathbf{x}). \tag{33}$$

Thus the reconstruction is a linear combination of the reconstructions according to Smith and Grangeat [$f_S(\mathbf{x})$ and $f_G(\mathbf{x})$, respectively], and the parameter γ controls the relative weighting between them. It is important to note that Eq. (1) and lemma 1 were not used to obtain Eq. (33). If $g(\boldsymbol{\theta},\lambda)$ were given by Eq. (1) then $f(\mathbf{x}) = f_S(\mathbf{x}) = f_G(\mathbf{x})$ and Eq. (33) would be trivial.

Further investigation of mathematical equivalence (or otherwise) of the inversion formulas for different γ is the subject of ongoing reserch.

APPENDIX A: FORMAL DERIVATION OF THE LEMMAS

Derivation of lemma 1: First note that from Eq. (3) we easily establish that for all positive scalars a, $h_1(at) = h_1(t)/a^2$, and that $h_2(at) = h_2(t)/a^2$. Furthermore, h_1 and h_2 are even and odd functions, respectively [i.e., $h_1(-t) = h_1(t)$ and $h_2(-t) = -h_2(t)$]. From Eqs. (1) and (2),

$$\begin{aligned} G(\boldsymbol{\beta},\lambda) &= \int_{S^2}\int_0^\infty f(\boldsymbol{\Phi}(\lambda)+t\boldsymbol{\theta})[ah_1+bh_2](\boldsymbol{\theta}\cdot\boldsymbol{\beta})\mathrm{d}t\mathrm{d}\boldsymbol{\theta} \\ &= \int_{S^2}\int_0^\infty f(\boldsymbol{\Phi}(\lambda)+t\boldsymbol{\theta})\frac{[ah_1+bh_2](\boldsymbol{\theta}\cdot\boldsymbol{\beta})}{t^2}t^2\mathrm{d}t\mathrm{d}\boldsymbol{\theta} \\ &= \int_{\mathbb{R}^3} f(\boldsymbol{\Phi}(\lambda)+\mathbf{x})[ah_1+bh_2](\mathbf{x}\cdot\boldsymbol{\beta})\mathrm{d}\mathbf{x} \\ &= \int_{\mathbb{R}^3} f(\mathbf{x})[ah_1+bh_2](\mathbf{x}\cdot\boldsymbol{\beta}-\boldsymbol{\Phi}(\lambda)\cdot\boldsymbol{\beta})\mathrm{d}\mathbf{x} \\ &= \int_{\mathbb{R}^3}\int_{-\infty}^{\infty} f(\mathbf{x})\delta(\mathbf{x}\cdot\boldsymbol{\beta}-t)[ah_1+bh_2](t-\boldsymbol{\Phi}(\lambda)\cdot\boldsymbol{\beta}) \\ &\quad \times \mathrm{d}t\mathrm{d}\mathbf{x} \\ &= \int_{-\infty}^{\infty} r(\boldsymbol{\beta},t)[ah_1+bh_2](t-l)|_{l=\boldsymbol{\Phi}(\lambda)\cdot\boldsymbol{\beta}}\mathrm{d}t \\ &= \int_{-\infty}^{\infty} r(\boldsymbol{\beta},t)[ah_1-bh_2](l-t)|_{l=\boldsymbol{\Phi}(\lambda)\cdot\boldsymbol{\beta}}\mathrm{d}t. \end{aligned} \tag{A1}$$

Therefore if $F(\boldsymbol{\beta},l)$ is given by Eq. (8) then $G(\boldsymbol{\beta},\lambda) = F(\boldsymbol{\beta},\boldsymbol{\Phi}(\lambda)\cdot\boldsymbol{\beta})$.

For the converse, first observe that if λ_1 and λ_2 are two solutions to $\boldsymbol{\Phi}(\lambda)\cdot\boldsymbol{\beta} = l$, then by Eq. (A1), $G(\boldsymbol{\beta},\lambda_1) = G(\boldsymbol{\beta},\lambda_2)$, and $F(\boldsymbol{\beta},l)$ can safely be defined as $F(\boldsymbol{\beta},\boldsymbol{\Phi}(\lambda)\cdot\boldsymbol{\beta}) = G(\boldsymbol{\beta},\lambda)$. Equation (8) then follows for all $\boldsymbol{\beta}$, and all $l \in \boldsymbol{\Phi}(\Lambda)\cdot\boldsymbol{\beta} = \{l:\boldsymbol{\Phi}(\lambda)\cdot\boldsymbol{\beta} = l\}$. By Kirillov's condition, $\boldsymbol{\Phi}(\Lambda)\cdot\boldsymbol{\beta} = (-\infty,\infty)$ and Eq. (8) follows for all $\boldsymbol{\beta}, l$. ∎

Derivation of lemma 2: Lemma 2 is essentially the classical 3D Radon inversion formula. Representing the 3D Fourier transform of f by $\check{f}$, the Fourier slice theorem in three dimensions states that

$$\check{f}(l\boldsymbol{\beta}) = \check{r}(\boldsymbol{\beta},l), \tag{A2}$$

where $\check{r}$ is the (one-dimensional) Fourier transform of r with respect to the second variable. Writing the inverse Fourier transform in spherical polar coordinates, and using Eq. (A2), we arrive at

$$\begin{aligned} f(x) &= \int_{S^2}\int_0^\infty \check{f}(\omega\boldsymbol{\beta})\exp(2\pi i\omega\boldsymbol{\beta}\cdot\mathbf{x})\omega^2\mathrm{d}\omega\mathrm{d}\boldsymbol{\beta} \\ &= \frac{1}{2}\int_{S^2}\int_{-\infty}^{\infty}\check{r}(\boldsymbol{\beta},\omega)\omega^2\exp(2\pi i\omega l)|_{l=\mathbf{x}\cdot\boldsymbol{\beta}}\mathrm{d}\omega\mathrm{d}\boldsymbol{\beta}, \end{aligned} \tag{A3}$$

which is the Radon inversion formula in 3D.

Writing ω^2 as $|\omega|\times|\omega|$ and applying the convolution theorem to the inner integral with the definition of h_1 from Eqs. (3), we arrive at Eq. (9). Writing ω^2 as $-1\times(i\omega)\times(i\omega)$ and using the definition of h_2, we also establish Eq. (10). ∎

Derivation of lemma 3: Expanding and collecting terms in $F^*(\boldsymbol{\beta},l)$ given by Eq. (11), we obtain

$$\begin{aligned} F^*(\boldsymbol{\beta},l) = {} & r(\boldsymbol{\beta},l) * [ac(h_1*h_1) + bd(-h_2*h_2)](l) \\ & + r(\boldsymbol{\beta},l) * [(ad-bc)(h_1*h_2)](l). \end{aligned} \tag{A4}$$

By lemma 2, Eq. (12) follows immediately if the second term in the above expression vanishes; i.e., we must establish that for all $\mathbf{x}$,

$$0 = \int_{S^2}[r(\boldsymbol{\beta},l)*h_1(l)*h_2(l)]_{l=\mathbf{x}\cdot\boldsymbol{\beta}}\mathrm{d}\boldsymbol{\beta}. \tag{A5}$$

Taking the Fourier transform with respect to l of the right-hand side (RHS) of this equation and applying the convolution theorem with the definitions of h_1 and h_2, we arrive at

$$\begin{aligned} \mathrm{RHS} &= \int_{S^2}\int_{-\infty}^{\infty}\check{r}(\boldsymbol{\beta},\omega)|\omega|(i\omega)\exp(2\pi i\omega l)|_{l=\mathbf{x}\cdot\boldsymbol{\beta}}\mathrm{d}\omega\mathrm{d}\boldsymbol{\beta} \\ &= \int_{S^2}\int_0^\infty \check{r}(\boldsymbol{\beta},\omega)i\omega^2\exp(2\pi i\omega\mathbf{x}\cdot\boldsymbol{\beta})\,\mathrm{d}\omega\mathrm{d}\boldsymbol{\beta} \\ &\quad + \int_{S^2}\int_{-\infty}^{0}\check{r}(\boldsymbol{\beta},\omega)(-i\omega^2)\exp(2\pi i\omega\mathbf{x}\cdot\boldsymbol{\beta})\,\mathrm{d}\omega\mathrm{d}\boldsymbol{\beta}. \end{aligned} \tag{A6}$$

Reversing the signs of ω and $\boldsymbol{\beta}$ in the second integral, we obtain

$$\mathrm{RHS} = \int_{S^2}\int_0^{\infty} \check{r}(\boldsymbol{\beta}, \omega) i\omega^2 \exp(2\pi i\omega \mathbf{x}\cdot\boldsymbol{\beta})\, \mathrm{d}\omega \mathrm{d}\boldsymbol{\beta} + \int_{S^2}\int_0^{\infty} \check{r}(-\boldsymbol{\beta}, -\omega)(-i\omega^2)\exp(2\pi i\omega \mathbf{x}\cdot\boldsymbol{\beta})\, \mathrm{d}\omega \mathrm{d}\boldsymbol{\beta}. \quad (A7)$$

Finally, because $\check{r}(-\boldsymbol{\beta}, -\omega) = \check{f}(\omega\boldsymbol{\beta}) = \check{r}(\boldsymbol{\beta}, \omega)$ by the Fourier slice theorem, the two terms cancel and RHS = 0 as required.

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